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Stability in the anisotropic Manev problem

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Abstract. The anisotropic Manev problem describes the motion of two point masses in an anisotropic space under the influence of a Newtonian force-law with a relativistic correction term. In this paper we prove the existence of a large, open and connected manifold of solutions of the planar anisotropic Manev problem that are uniformly bounded and collisionless.

1. Introduction

In the mid 1990s we proposed the anisotropic Manev problem in order to find relationships connecting classical, quantum and relativistic mechanics. This was possible due to the presence of the relativistic correction term of the Manev potential, which we added to the classical Newtonian one (see [8]). The correction term is absent from the anisotropic Kepler problem, previously studied for the purpose of understanding the similarities between classical and quantum mechanics (see [1, 4, 5, 10–13]).

In an earlier paper we showed, among other things, that solutions with classical, relativistic and quantum features coexist in the anisotropic Manev problem (see [2]). However, all these solutions lead to nonregularizable collisions, i.e. to singularities beyond which the orbit cannot be meaningfully extended (see [6]). Though crucial for understanding the dynamical behaviour of solutions and important for showing that this problem lies at the intersection of the above-mentioned fields, collision orbits cannot provide positive answers to stability questions. Since stability is relevant to both mathematics and physics, our next goal is to understand whether we encounter it in the anisotropic Manev problem.

In this paper we report on a preliminary result. In what follows we will show evidence of a type of weak stability by proving the existence of a large, open and connected manifold of uniformly bounded, collisionless solutions. The future study of the solutions in this set may reveal some connections between the above-mentioned branches of physics.

This paper is structured as follows. In section 2 we introduce the equations of motion, bring some arguments that support the study of this model, and briefly describe the results obtained up to now. In section 3 we perform some McGehee-type transformations (see [14]), which lead us to an equivalent system of differential equations, suitable for our purposes. Finally, in section 4, we prove our stability result and draw some final conclusions.

2. Equations of motion

We consider the two-degrees-of-freedom Hamiltonian system

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{p} \\ \dot{\boldsymbol{p}} &= \nabla W(\boldsymbol{q}) \end{aligned} \tag{1}$$

where $\boldsymbol{q} = (q_1, q_2)$ and $\boldsymbol{p} = (p_1, p_2)$ denote the configuration and the momentum of a planar physical system of two point masses, with one of the particles fixed at the origin of the reference frame. The real analytic function $W: \mathbb{R} \setminus \{\mathbf{0}\} \to (0, \infty)$ is a quasihomogeneous potential (i.e. a sum of two homogeneous functions, which is not necessarily homogeneous), given by

$$W(\boldsymbol{q}) = \frac{1}{\sqrt{\mu q_1^2 + q_2^2}} + \frac{b}{\mu q_1^2 + q_2^2}$$

where $\mu > 1$ and $b > 0$ (small) are parameters. The constant $\mu > 1$ defines the anisotropy of the space. For $\mu = 1$ we recover the classical (isotropic) Manev problem, which has been studied starting with Newton (see [3] and [9] for its complete qualitative understanding). The small constant b offers a relativistic correction to the classical gravitational law (see [9]). We introduced the study of quasihomogeneous potentials at the beginning of the 1990s (see [7,8]) in order to provide a unifying mathematical framework for several potentials used in astronomy, physics and chemistry, as for example the ones of Newton, Coulomb, Manev, Birkhoff, Schwarzschild, van der Waals, Liboff, Lennard-Jones and others. The anisotropic Manev problem is only one direction of work in this more general context. In addition, Manev-type potentials play a special role within the class of quasihomogeneous potentials, as we have shown in [8].

The Hamiltonian function of system (1) is

$$H(\boldsymbol{p}(t), \boldsymbol{q}(t)) = \tfrac{1}{2}|\boldsymbol{p}(t)|^2 - W(\boldsymbol{q}(t))$$

which represents the sum of the kinetic energy $\frac{1}{2}|\boldsymbol{p}|^2$ and the potential energy $-W(\boldsymbol{q})$. This yields the energy integral

$$H(\boldsymbol{p}(t), \boldsymbol{q}(t)) = h \tag{2}$$

where h is a real constant. However, the angular momentum $L(t) = \boldsymbol{p}(t) \times \boldsymbol{q}(t)$ does not lead to an integral of the system, as it occurs in the classical (isotropic) Manev problem (see [3]).

Since W is real analytic, standard results of differential-equations theory ensure that for any initial data $(\boldsymbol{q}(0), \boldsymbol{p}(0)) \in (\mathbb{R} \setminus \{\mathbf{0}\}) \times \mathbb{R}$ there exists a unique analytic solution defined on a maximal interval $[0, t^*)$, where $0 < t^* \leqslant \infty$. If $t^* < \infty$, the solution is said to experience a singularity. This occurs when $\boldsymbol{q}(t) \to \mathbf{0}$ as $t \to t^*$. We have shown that in the anisotropic Manev problem all singularities are collisions (see [2,8]), a result that is false in the Newtonian n-body problem for $n > 4$ (see [6]).

In [2] we mainly studied the behaviour of solutions close to collisions and also obtained some properties of the zero-energy flow. Among other things we proved that the set of initial conditions leading to collisions has positive measure and that heteroclinic orbits exist in the zero-energy case. These are strong arguments favouring the nonintegrability of the system. However, nonintregrability does not necessarily exclude some form of stability, as for example the one in the sense of Poisson.

In what follows we will prove the existence of a large, open and connected manifold of bounded and collisionless solutions. This shows that the system has an invariant manifold that maintains some degree of stability. Since the isotropic Manev problem also possesses such a set (formed by periodic and quasiperiodic orbits—see [3,9]) it means that the anisotropy does not entirely destroy this set. We do not know whether all or only some periodic orbits of the isotropic problem survive the anisotropic impact, but we will show that many of the bounded and collisionless solutions of the isotropic problem remain bounded and collisionless in the anisotropic one.

3. Transformations

In order to prove our stability result, we first need to express the equations of motion (1) in terms of some more convenient variables. For this we consider the coordinate transformations

$$\begin{aligned} r &= |\boldsymbol{q}| \\ \theta &= \arctan(q_2/q_1) \\ y &= \dot{r} = (q_1 p_1 + q_2 p_2)/|\boldsymbol{q}| \\ x &= r\dot{\theta} = (q_1 p_2 - q_2 p_1)/|\boldsymbol{q}| \end{aligned}$$

then the rescaling change of variables

$$\begin{aligned} v &= ry \\ u &= rx \end{aligned}$$

and finally the time transformation

$$\mathrm{d}\tau = r^{-1}\,\mathrm{d}t. \tag{3}$$

Similar transformations were introduced in 1974 by Richard McGehee (see [14]) in the study of the Newtonian rectilinear three-body problem. Note that r and θ are polar coordinates, whereas y and x represent the radial and tangential components of the velocity. The variables v and u rescale y and x with the help of r, and τ is the new independent variable that rescales the time.

Composing these transformations, which are analytic diffeomorphisms in their respective domains, system (1) becomes

$$\begin{aligned} r' &= v \\ v' &= 2hr + (\mu\cos^2\theta + \sin^2\theta)^{-1/2} \\ \theta' &= r^{-1}u \\ u' &= \frac{\mu-1}{2} r^{-1}[(\mu\cos^2\theta + \sin^2\theta)^{-3/2} + 2b(\mu\cos^2\theta + \sin^2\theta)^{-2}]\sin 2\theta \end{aligned} \tag{4}$$

and the energy relation (2) takes the form

$$u^2 + v^2 - 2r(\mu\cos^2\theta + \sin^2\theta)^{-1/2} - 2b(\mu\cos^2\theta + \sin^2\theta)^{-1} = 2hr^2. \tag{5}$$

The new dependent variables $(r, v, \theta, u) \in (0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}$ are functions of the fictitious time τ, so from now on the prime denotes differentiation with respect to it. System (4) is equivalent to system (1).

Note that system (4) may still have singularities at $r = 0$, i.e. when collisions take place. We could have removed them by considering, instead of (3), the time transformation $\mathrm{d}\tau = r^{-2}\,\mathrm{d}t$. The corresponding system would have been suitable for studying collision and near-collision orbits, as was initiated in [14]. However, since we are now interested in collisionless solutions, we prefer to work with system (4), which will lead us to the desired result.

4. Stability

We are now in the position to prove the main result of this paper. We will show that there exists an open, connected invariant manifold of solutions corresponding to system (4) for which the variable r is always uniformly bounded from above and larger than a positive constant. Since r is a measure of the distance between particles, it means that every solution in this set is bounded and collisionless. Let us formally state this property as follows.

Theorem. *The negative-energy manifold of solutions* (r, v, θ, u) *of system (4) contains an open, nonempty and connected submanifold in which all solutions are uniformly bounded and collisionless.*

Proof. Let us first note that there exist two constants, m_1 and m_2 with $0 < m_1 < m_2 < \infty$, such that along any solution (r, v, θ, u) of system (4), the function of τ given by $\mu \cos^2\theta + \sin^2\theta$ satisfies the relations

$$m_1 < \mu \cos^2\theta + \sin^2\theta < m_2$$

for all τ for which $\theta(\tau)$ is defined. Consequently there exist two constants, M_1 and M_2 with $0 < M_1 < M_2 < \infty$, such that

$$M_1 < (\mu \cos^2\theta + \sin^2\theta)^{-1/2} < M_2 \tag{6}$$

for all τ for which θ is defined. □

From the first two equations of system (4) we obtain the nonhomogeneous second-order equation

$$r'' - 2hr = (\mu \cos^2\theta + \sin^2\theta)^{-1/2} \tag{7}$$

which represents a forced harmonic oscillator. We will further ignore the last two equations of system (4) and retain only the information that the forcing function $(\mu \cos^2\theta + \sin^2\theta)^{-1/2}$ is uniformly bounded from above and away from zero from below, as relations (6) show.

We will further restrict our analysis to the invariant manifold of solutions of negative energy, $h < 0$, which obviously exists according to relation (5). Solving the homogeneous equation $r'' - 2rh = 0$ and then applying the method of variation of parameters to equation (7), we obtain the general solution of equation (7) in the form

$$\begin{aligned} r(\tau) = &\left(c_1 + \frac{1}{\sqrt{-2h}} \int_0^\tau \frac{\cos\left(\sqrt{-2h}\sigma\right)}{\sqrt{\mu \cos^2\theta(\sigma) + \sin^2\theta(\sigma)}}\, \mathrm{d}\sigma\right) \sin\left(\sqrt{-2h}\tau\right) \\ &+\left(c_2 - \frac{1}{\sqrt{-2h}} \int_0^\tau \frac{\sin\left(\sqrt{-2h}\sigma\right)}{\sqrt{\mu \cos^2\theta(\sigma) + \sin^2\theta(\sigma)}}\, \mathrm{d}\sigma\right) \cos\left(\sqrt{-2h}\tau\right). \end{aligned} \tag{8}$$

If, for every solution of constants c_1 and c_2, we apply to the above integrals the intermediate-value theorem, we obtain

$$\begin{aligned} r(\tau) = c_1 \sin\left(\sqrt{-2h}\tau\right) + c_2 \cos\left(\sqrt{-2h}\tau\right) &- \frac{\sin^2\left(\sqrt{-2h}\tau\right)}{2h\sqrt{\mu \cos^2\theta(\tau_1) + \sin^2\theta(\tau_1)}} \\ &- \frac{\cos^2\left(\sqrt{-2h}\tau\right)}{2h\sqrt{\mu \cos^2\theta(\tau_2) + \sin^2\theta(\tau_2)}} \end{aligned}$$

where τ_1 and τ_2 belong to the interval $(0, \tau)$. Using some trigonometry, this estimate can be written as

$$r(\tau) = \sqrt{c_1^2 + c_2^2}\cos(\omega\tau - \omega_0) - \frac{\sin^2\left(\sqrt{-2h}\tau\right)}{2h\sqrt{\mu\cos^2\theta(\tau_1) + \sin^2\theta(\tau_1)}} - \frac{\cos^2\left(\sqrt{-2h}\tau\right)}{2h\sqrt{\mu\cos^2\theta(\tau_2) + \sin^2\theta(\tau_2)}} \tag{9}$$

where $\omega = \sqrt{-2h}$ and $\omega_0 = \arctan\frac{c_2}{c_1}$. Relations (6) and (9) allow us now to draw the conclusion that for any solution (r, v, θ, u) of system (4), the inequalities

$$\sqrt{c_1^2 + c_2^2}\cos(\omega\tau - \omega_0) - \frac{M_1}{2h} < r(\tau) < \sqrt{c_1^2 + c_2^2}\cos(\omega\tau - \omega_0) - \frac{M_2}{2h} \tag{10}$$

take place for all τ for which the solution is defined. (The domain of each solution depends on whether collisions take place; if no collisions occur, then the solution is defined on $\mathbb{R}$, otherwise it is defined on some open interval of which at least one end is finite. Also note that a finite-end interval could in principle occur if the solution encounters a noncollision singularity, but as we have proved in [2]—see also section 2 of this paper—the anisotropic Manev problem is free of such singularities.)

Let us now fix some $h_0 < 0$, as close to 0 as we like, and define the set Λ of solutions formed by the union of all sets Λ_h, for h in the interval $(-\infty, h_0)$, where each set Λ_h contains all the solutions given by (8) that satisfy the inequality

$$c_1^2 + c_2^2 < \frac{M_1^2}{4h^2}.$$

Obviously, this is an open, nonempty and connected invariant manifold of system (4). From (10) it follows that, for every solution of this manifold, r is positive and bounded, therefore the orbits are collisionless and bounded. The smaller the constant b (and physical applications require b to be small, of the order of 10^{-10} or less), the larger the above-defined manifold. This completes the proof.

In conclusion, we have proved the existence of a large, open and connected manifold of solutions for a system of differential equations describing the anisotropic Manev problem, which lies at the intersection of classical, quantum and relativistic mechanics. Understanding the dynamics of this set may reveal some connections between these fundamental branches of physics.

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References

[1] Casasayas J and Llibre J 1984 Qualitative analysis of the anisotropic Kepler problem *Memoirs of the American Mathematical Society* vol 52 (Providence, RI: American Mathematical Society) p 312

[2] Craig S, Diacu F, Lacomba E A and Perez E 1999 On the anisotropic Manev problem *J. Math. Phys.* **40** 1359–75

[3] Delgado J, Diacu F, Lacomba E A, Mingarelli A, Mioc V, Perez E and Stoica C 1996 The global flow of the Manev problem *J. Math. Phys.* **37** 2748–61

[4] Devaney R L 1978 Collision orbits in the anisotropic Kepler problem *Inventiones Math.* **45** 221–51

[5] Devaney R L 1981 Singularities in classical mechanical systems *Ergodic Theory and Dynamical Systems* vol 1 (Boston: Birkhäuser) pp 211–333

[6] Diacu F 1992 *Singularities of the N-Body Problem—An Introduction to Celestial Mechanics* (Montréal: Les Publications CRM)

[7] Diacu F 1993 The planar isosceles problem for Maneff's gravitational law *J. Math. Phys.* **34** 5671–90

[8] Diacu F 1996 Near-collision dynamics for particle systems given by quasihomogeneous potentials *J. Diff. Eqns* **128** 58–77

[9] Diacu F, Mioc V and Stoica C 2000 Phase-space structure and regularization of Manev-type problems *Nonlinear Anal.* **41** 1029–55

[10] Gutzwiller M C 1971 Periodic orbits and classical quantization conditions *J. Math. Phys.* **12** 343–58

[11] Gutzwiller M C 1973 The anisotropic Kepler problem in two-dimensions *J. Math. Phys.* **14** 139–52

[12] Gutzwiller M C 1977 Bernoulli sequences and trajectories in the anisotropic Kepler problem *J. Math. Phys.* **18** 806–23

[13] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)

[14] McGehee R 1974 Triple collision in the collinear three-body problem *Inventiones Math.* **27** 191–227